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Asymptotic Solution for Viscous, Absorbing, Emitting Shock Layer

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Absorbing, emitting, high Reynolds number flow in the blunt body stagnation region is studied using asymptotic expansions. In contrast to previous results for optically thin gases, the boundary layer is here shown to be radiating and scale as $Re^{-1/2}$ with corrections of order $(\ln Re)^{1+j}Re^{-1/2}$. Typical parameter values suggest the boundary layer to be weakly radiating. The primary effect of radiative transfer in the inviscid portion of the shock layer on the boundary-layer flow is given by the reduced value of the inviscid gas enthalpy at the wall. Numerical calculations of skin friction and total heat transfer including both radiation-induced coupling of the radiation and convective components and contributions of higher-order corrections indicate reduction of the skin friction and convective heat transfer due to radiative transfer. The results are used to determine conditions for which the radiative and convective components of heat transfer are equal.

Nomenclature

a_i = constants
A = reference area
B_ν = Planck function
$\bar{B}_\nu$ = nondimensional Planck function
c = speed of light
C_f = skin-friction coefficient
E_n = nth order exponential integral function
f = reduced stream function
F = reduced stream function in boundary layer
$\bar{F}$ = $\partial F/\partial \bar{\Gamma}$
h = enthalpy
H = boundary-layer enthalpy
$\bar{H}$ = $\partial H/\partial \bar{\Gamma}$
j = 0,1 for two-dimensional and axisymmetric flow, respectively
L = shock layer thickness scale
m = exponent in variation of density with enthalpy
N = $\rho\mu/\rho_s\mu_s$
p = pressure
Pr = Prandtl number
q^R = radiative heat flux
q^c = convective heat flux
r = radius from axis of symmetry
Re = Reynolds number
T = temperature
u,v = velocity components in the x and y directions, respectively
x,y = coordinates parallel to and normal to the body, respectively
y^* = value of y in overlap region
α = exponent in variation of temperature with enthalpy
β = exponent in variation of mass absorption coefficient with temperature
γ = Euler constant, 0.577...
Γ = radiation-convection energy parameter
$\bar{\Gamma}$ = boundary-layer radiation-convection energy parameter
δ_1,δ_2 = small parameters of boundary-layer expansion
ϵ_1 = small parameter of inviscid layer expansion
ζ = boundary-layer coordinate
η = normalized optical variable
$\bar{\eta}$ = boundary-layer optical variable
κ = mass absorption coefficient
κ_p = Planck mean mass absorption coefficient
μ = viscosity coefficient
ν = frequency
π^2 = pressure gradient parameter
ρ = density
σ = Stefan-Boltzmann constant
τ_L = shock layer optical depth
τ_s = reduced shock layer optical depth
τ_ν = optical depth at frequency ν
$\bar{\tau}_s$ = boundary-layer optical depth
φ_i = boundary-layer scaling parameters
ψ = stream function

Subscripts

ν = frequency dependent
w = wall
s = shock
0,1,2 = lowest, first, second order
thin = result obtained from optically thin analysis

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Superscripts

c = composite
i = inner
o = outer

Introduction

THE importance of radiative energy transfer in high-speed re-entry is well known. With the exception of the work of Burggraf,[1] previous efforts to calculate the heat transfer for high Reynolds number flow past a blunt body have either assumed the flow to be inviscid,[2,3] neglecting the convective component entirely, or have, within the framework of purely numerical methods, taken the shock layer to be fully viscous,[4,5] without consideration of the inviscid-viscous division of the shock layer at high Reynolds number. The inviscid approximation is presumably a lowest-order approximation, valid outside the viscous boundary layer near the body. Numerical schemes for the fully viscous shock layer while valid at low Reynolds numbers[6,7] cannot be extended to arbitrary high Reynolds numbers due to the singular nature of that limit. The asymptotic results of Burggraf[1] for high Reynolds number flow, while including both convective and radiative components, are restricted to optically thin shock layers and do not yield the usual inverse-square-root dependence of boundary-layer thickness on Reynolds number. In view of the recently established singular nature of the optically thin limit,[8] it would appear that this result is of limited validity. The removal of the optically thin approximation with the inclusion of self-absorption effects, while of obvious importance, implies conservation equations of a nonlinear integro-differential form, analytical solutions of which are quite difficult. Thus, for values of the radiation-convection $(\sigma T^4/puh)_{\text{ref}}$, Bouguer $(\varkappa L)_{\text{ref}}$, and Reynolds $(\rho UL/\mu)_{\text{ref}}$ numbers typical of very high-speed re-entry, calculations of the total heat transfer in the boundary-layer limit, including coupling of the radiative and convective components, are presently unavailable.

With this in mind, the investigation to be described has had the following objectives: 1) application of the method of matched asymptotic expansions to the high Reynolds number flow of a radiating gas in the stagnation region of a blunt body, including self-absorption effects, with elucidation of the boundary-layer structure; 2) calculation of the total heat transfer including both coupling of the radiative and convective components and the contribution of higher-order corrections; 3) detailed numerical solutions of the flowfield near an axisymmetric stagnation point for typical values of the relevant parameters; and 4) comparison with previously obtained results for the optically thin limit so as to ascertain the validity of those results.

Governing Equations

Consider flow of a viscous, radiating gas near the stagnation point of a blunt body. Taking x and y parallel and normal to the body, respectively, conservation of mass, Newton's second law, and conservation of energy are

$$(\partial/\partial x)(\rho ur^j) + (\partial/\partial y)(\rho vr^j) = 0 \tag{1}$$

$$\rho u\partial u/\partial x + \rho v\partial u/\partial y + \partial p/\partial x = (\partial/\partial y)(\mu\partial u/\partial y) \tag{2}$$

$$-\rho u^2/r + \partial p/\partial y = 0 \tag{3}$$

$$\rho u\partial h/\partial x + \rho v\partial h/\partial y + \nabla\cdot\mathbf{q}^R = (\partial/\partial y)[(\mu/Pr)\partial h/\partial y] \tag{4}$$

That is, we accept the thin shock layer equations as being the appropriate descriptive equations in the stagnation region. Note that the dissipation terms do not appear as they are negligible near the stagnation point. As shown by Bush[9] in the nonradiating case, the above equations along with a discontinuous bow shock wave model (e.g. the ordinary Rankine-Hugoniot jump conditions apply across the shock discontinuity) provide a proper representation of the shock layer up to (but not including) order Re^{-1}. One does not expect this result to be altered by the inclusion of radiative energy transfer. Also, the radiative pressure and energy density have been neglected as they are unimportant for problems of interest. Following Burggraf,[1] we shall, for purposes of simplicity, assume the pressure gradient $\partial p/\partial x$ to be a known function of x as in the boundary-layer theory and thus shall neglect the variation of the pressure across the shock layer.

Using the tangent slab approximation and neglecting the effect of upstream absorption, the divergence of the radiative heat flux, $\nabla\cdot\mathbf{q}^R$, is given by

$$\nabla\cdot\mathbf{q}^R = 4\pi\int_0^\infty \rho\varkappa_\nu B_\nu d\nu - 2\pi\int_0^\infty \rho\varkappa_\nu B_\nu(T_w)E_2(T_\nu)d\nu - 2\pi\int_0^\infty \rho\varkappa_\nu\int_0^{\tau_{\nu s}} B_\nu(t)E_1(|\tau_\nu - t|)dtd\nu \tag{5}$$

assuming local thermodynamic equilibrium. Here $\varkappa_\nu$ is the mass absorption coefficient which depends, in general, upon frequency as well as the thermodynamic state. B_ν is the Planck function and E_n is the nth-order exponential integral function. The optical depth τ_ν is given by

$$\tau_\nu = \int_0^y \rho\varkappa_\nu dy \tag{6}$$

In writing Eq. (5), we have assumed the body to be black at a temperature T_w.

We introduce the following nondimensional variables

$$\bar{x} = \frac{x}{L},\ \bar{y} = \frac{1}{L}\int_0^y \frac{\rho}{\rho_s}dy,\ \bar{\varkappa} = \frac{\varkappa}{\varkappa_s}$$
$$\bar{\rho} = \rho/\rho_s,\ \bar{u} = u/u_s,\ \bar{v} = \rho v/\rho_s v_s \tag{7}$$
$$\bar{h} = h/h_s,\quad \bar{p} = p/\rho_s v_s^2,\quad \bar{T} = T/T_s$$

For simplicity, we assume $\varkappa_\nu$, the mass absorption coefficient, to have an "appropriate" frequency-averaged value $\varkappa$ allowing use of the gray-gas approximation. While qualitative behavior is predicted well by the gray-gas model, quantitative errors can arise and this should be kept in mind when examining the results. Later, we shall comment on the effect of including nongrey behavior. The following approximate, but reasonable, relations between the density, temperature, mass absorption coefficient, and enthalpy will be employed

$$\rho = h^{-m},\quad T^4 = h^\alpha,\quad \varkappa = h^\beta \tag{8}$$

We define the stream function ψ by

$$\partial\psi/\partial y = ux^j,\quad \partial\psi/\partial x = -vx^j \tag{9}$$

The continuity equation is satisfied identically if the stream function ψ is given by

$$\psi = (x^{1+j}/1 + j)f(y) \tag{10}$$

Taking the enthalpy in the stagnation region to be a function of y alone, we can then write Eqs. (2) and (4) as, dropping the bar notation,

$$(1/Re)(Nf_{yy})_y + ff_{yy} = (f^2 - \pi^2h^m)/(1 + j) \tag{11}$$

$$\frac{1}{Re}\left(\frac{N}{Pr}h_y\right)_y + fh_y = \Gamma h^\beta \times \left[2h^\alpha - T_w{}^4E_2(\tau_s\eta) - \tau_s\int_0^1 h^\alpha E_1(\tau_s|\eta - \eta'|)d\eta\right] \tag{12}$$

Here the subscript y implies differentiation with respect to that variable. The normalized optical variable η is given by

$$\eta = \int_0^y \varkappa dy\Big/\int_0^1 \varkappa dy \tag{13}$$

with τ_s being the shock layer optical depth

$$\tau_s = \rho_s\kappa_s L\int_0^1 \varkappa dy = \tau_L\int_0^1 \varkappa dy \tag{14}$$

The Reynolds number Re, the Prandtl number Pr, the radiation-convection energy parameter (inverse Boltzmann number) Γ, and N are given by

$$Re = \rho_s v_s L/\mu_s, \quad Pr = \mu C_p/k$$
$$\Gamma = 2\sigma T_s \rho_s \varkappa_s L/\rho_s v_s h_s, \; N = \rho\mu/\rho_s\mu_s \tag{15}$$

Also, π^2 is a pressure gradient parameter, or Euler number

$$\pi^2 = -[(1+j)^2/x]dp/dx \tag{16}$$

typically bounded by 0.1 and 1.0 for conditions of interest. The boundary conditions for Eqs. (11) and (12) are, neglecting possible mass transfer at the wall,

$$y = 1 \text{ (shock)}: \quad f = h = 1$$
$$y = 0 \text{ (wall)}: \quad f = f_y = 0 \tag{17}$$
$$h = h_w = h(T_w)$$

assuming the wall temperature is given.

Inviscid Flow

As we are concerned with the high Reynolds number limit, it is appropriate to consider an expansion about infinite Reynolds number with y fixed corresponding to the inviscid outer flow. It is well known that such an outer expansion procedure is singular with the region of invalidity being near the wall ($y = 0$). Thus we let

$$f^o = f_0{}^o + \epsilon_1(Re)f_1{}^o + \ldots$$
$$h^o = h_0{}^o + \epsilon_1(Re)h_1{}^o + \ldots \tag{18}$$

where here the superscript o means outer and ϵ_1 is small for large Re. Substituting these expansions into Eqs. (11) and (12), we obtain, to lowest order

$$f_0{}^o f_{0_{yy}}{}^o = (f_{0_y}{}^{o2} - \pi^2 h_0{}^{om})/(1+j) \tag{19}$$

$$f_0{}^o h_{0_y}{}^o = \Gamma h_0{}^{o\beta}\Big[2h_0{}^{o\alpha} - T_w{}^4 E_2(\tau_s\eta_0) - \tau_s\int_0^1 h_0{}^{o\alpha}E_1(\tau_s|\eta_0 - \eta_0'|)d\eta_0'\Big] \tag{20}$$

with boundary conditions at the bow shock wave

$$y = 1, \quad h_0{}^o = f_0{}^o = 1 \tag{21}$$

The boundary conditions at the wall($y = 0$) follow from the matching. Equations (19) and (20) have been derived under the assumption

$$Re\epsilon_1(Re) \to \infty \text{ as } Re \to \infty \tag{22}$$

which will be verified later.

To determine the first order equations, we must consider the absorption integral in some detail. Consider then

$$I = \tau_L\int_0^1 h^{\alpha+\beta}E_1\Big(\tau_L\int_y^{y'} h^\beta dy''\Big)dy' \tag{23}$$

We now assume that an asymptotically correct value of this integral follows upon use of the composite solution in the integrand. Thus we can write

$$I \sim \tau_L\int_0^1 [h_0{}^{c\alpha+\beta} + \epsilon_1(\alpha+\beta)h_1{}^c h_0{}^{c\alpha+\beta-1} + \ldots]\cdot E_1\Big(\tau_L\Big|\int_y^{y'}(h_0{}^{c\beta} + \beta\epsilon_1 h_1{}^c h_0{}^{c\beta-1} + \ldots)dy''\Big|\Big)dy' \tag{24}$$

Here the superscript c means composite solution, taken to be

$$h^c = h^o + h^i - h^{io} \tag{25}$$

with the superscript i meaning inner, boundary-layer solution. Taking advantage of the result that the enthalpy in the inner region scales as $h_{0_{wg}}{}^o$ (to be shown later), we can rewrite this equation as

$$h^c = h^o + h_{0_{wg}}{}^o H^i - h^{io} \tag{26}$$

Thus H^i here is order unity. Noting that the composite solution is equal to the outer solution everywhere except in the inner region, we can write

$$I \sim \tau_L\int_0^1 [h_0{}^{o\alpha+\beta} + \epsilon_1(\alpha+\beta)h_1{}^o h_0{}^{o\alpha+\beta-1} + \ldots]\cdot E_1\Big(\tau_L\Big|\int_y^{y'} h_0{}^{o\beta} + \epsilon_1\beta h_1{}^o h_0{}^{o\beta-1} + \ldots dy''\Big|\Big)dy' + \tau_L\int_0^{y^*}\Big[h^{c\alpha+\beta}E_1\Big(\tau_L\Big|\int_y^{y'} h^{c\beta}dy''\Big|\Big) - h^{o\alpha+\beta}E_1\Big(\tau_L\Big|\int_y^{y'} h^{o\beta}dy''\Big|\Big)\Big]dy' \tag{27}$$

where y^* corresponds to a value of y in the overlap region. Letting $\zeta = y\varphi_1$ ($\varphi_1 \gg 1$) of order unity describe the inner region (φ_1 to be determined later) we can rewrite the second term as

$$\frac{\tau_L}{\varphi_1}h_{0_{wg}}{}^{o\alpha+\beta}\int_0^{y^*\varphi_1}\Big[H_0{}^{i\alpha+\beta}E_1\Big(\tau_L\Big|\int_{\zeta'/\varphi_1}^{y} h^{c\beta}dy''\Big|\Big) - E_1\Big(\tau_L\Big|\int_{\zeta'/\varphi_1}^{y} h^{o\beta}dy''\Big|\Big)\Big]d\zeta'$$

where we have substituted the inner solution for the composite solution in the inner region. Expanding this result for large φ_1, we have, to lowest order

$$\frac{\tau_L}{\varphi_1}h_{0_{wg}}{}^{o\alpha+\beta}E_1\Big(\tau_L\int_0^y h_0{}^{o\beta}dy'\Big)\int_0^\infty (H_0{}^{i\alpha+\beta} - 1)d\zeta'$$

Expanding the other terms in the absorption integral for small ϵ_1, we have

$$I \sim \tau_L\int_0^1 h_0{}^{o\alpha+\beta}E_1\Big(\tau_L\Big|\int_y^{y'} h_0{}^{o\beta}dy''\Big|\Big)dy' + \epsilon_1\Big[\tau_L(\alpha+\beta)\int_0^1 h_0{}^{o\alpha+\beta-1}h_1{}^o E_1\Big(\tau_L\Big|\int_y^{y'} h_0{}^{o\beta}dy''\Big|\Big)dy' - \tau_L{}^2\beta\int_0^1 h_0{}^{o\alpha+\beta}E_0\Big(\tau_L\Big|\int_y^{y'} h_0{}^{o\beta}dy''\Big|\Big)\Big|\int_{y'}^{y} h_0{}^{o\beta-1}h_1{}^o dy''\Big|dy'\Big] - \frac{\tau_L}{\varphi_1}h_{0_{wg}}{}^{o\alpha+\beta}E_1(\tau_s\eta_0)\int_0^\infty (1 - H_0{}^{i\alpha+\beta})d\zeta' + \ldots \tag{28}$$

It should be noted that the expansion of

$$E_1\Big(\tau_L\Big|\int_{y'}^{y} h^{o\beta}dy''\Big|\Big)$$

given above is not uniformly valid. Specifically, it is not valid near $y = 0$ as $h_1{}^o$ is logarithmically singular there. However, as the region of invalidity is exponentially small, it is completely immersed in the algebraically small boundary layer and therefore is neglected. The first-order inviscid equations can thus be written

$$f_0{}^o f_{1_{yy}}{}^o + f_1{}^o f_{0_{yy}}{}^o = (2f_{0_y}{}^o f_{1_y}{}^o - \pi^2 m h_0{}^{om-1}h_1{}^o)/(1+j) \tag{29}$$

$$f_0{}^o h_{1_y}{}^o + f_1{}^o h_{0_y}{}^o = \beta h_1{}^o f_0{}^o h_{0_y}{}^o/h_0{}^o + \Gamma h_0{}^{o\beta}\Big[2\alpha h_0{}^{o\alpha-1}h_1{}^o + \tau_s\eta_1 T_w{}^4 E_1(\tau_s\eta_0) - \tau_s(\alpha+\beta)\int_0^1 h_0{}^{o\alpha-1}h_1{}^o E_1(\tau_s|\eta_0-\eta_0'|)d\eta_0' + \tau_s{}^2\beta\int h_0{}^{o\alpha}E_0(\tau_s|\eta_0-\eta_0'|)|\eta_1-\eta_1'|d\eta_0' - \tau_L h_{0_{wg}}{}^{o\alpha+\beta}E_1(\tau_s\eta_0)\int_0^\infty (1 - H_0{}^{i\alpha+\beta})d\zeta'\Big] \tag{30}$$

assuming $\epsilon_1 = 1/\varphi_1$, a result verified later. Here

$$\eta_0 = \int_0^y h_0{}^{o\beta}dy'\Big/\int_0^1 h_0{}^{o\beta}dy', \quad \eta_1 = \beta\int_0^{\eta_0} h_1{}^o/h_0{}^o d\eta_0' \tag{31}$$

The boundary conditions at the bow shock wave are

$$y = 1: \quad h_1{}^o = f_1{}^o = 0 \tag{32}$$

Taking advantage of the intuitively apparent result (verified in detail later) that the lowest-order, outer streamfunction $f_0{}^o$, vanishes at the body, one can show from Eqs. (19, 20, 29, 30) that the inner limit of the outer solution, needed for the matching, is given by

$$f \sim \pi h_{0_{wg}}{}^{o^{m/2}} y + a_1 y^2 (\log y)^{1+j} + a_2 y^2 (\log y)^j + \epsilon_1 [f_{1_{wg}}{}^o + a_3 y (\log y)^{1+j}] + o(\epsilon_1 y (\log y)^j, y^2) \tag{33}$$

$$h \sim h_{0_{wg}}{}^o + a_4 (y \log y - y) + a_5 y + \epsilon_1 (a_6 \log y + a_7) + o(\epsilon_1 y_1 y^2) \tag{34}$$

where $h_{0_{wg}}{}^o$, the lowest-order inviscid gas enthalpy at the wall, is given by

$$h_{0_{wg}}{}^o = \left[\frac{1}{2}\left(T_w{}^4 + \tau_s \int_0^1 h_0{}^{o\alpha} E_1(\tau_s \eta') d\eta'\right)\right]^{1/\alpha} \tag{35}$$

and follows simply from the condition of radiative equilibrium at $y = 0$. The a_i are given in Appendix A.

Boundary-Layer Flow

As is well known, a regular perturbation expansion about the infinite Reynolds number limit is not uniformly valid. As the region of invalidity is near the wall ($y = 0$), we introduce the stretched inner coordinate ζ and inner variables H^i and F^i as

$$\zeta = y\varphi_1, \quad H^i = h\varphi_2, \quad F^i = f\varphi_3 \tag{36}$$

where φ^i are constants, depending upon the Reynolds number and other parameters, to be determined in such a manner that near the wall where momentum and energy transfer by transport phenomena are important, ζ, H^i, and F^i are of order unity. Using these inner variables, we can write Eqs. (11) and (12) as

$$(NF_{\zeta\zeta}{}^i)_\zeta + \frac{Re}{\varphi_1\varphi_3} F^i F_{\zeta\zeta}{}^i = \frac{Re}{\varphi_1\varphi_3}\frac{1}{1+j}\left(F_\zeta{}^{i^2} - \frac{\pi^2\varphi_3{}^2}{\varphi_1{}^2\varphi_2{}^m} h^{i^m}\right) \tag{37}$$

$$(NH_\zeta{}^i)_\zeta + \frac{Re}{\varphi_1\varphi_3} F^i H_\zeta{}^i = \frac{\Gamma Re}{\varphi_1{}^2 \varphi_2{}^{\alpha+\beta-1}} \times H^{i^\beta}\left[2 H^{i^\alpha} - T_w{}^4 \varphi_2{}^\alpha E_2\left(\frac{\tau_s}{\varphi_1\varphi_2{}^\beta}\int_0^\zeta H^{i^\beta} d\zeta \Big/ \int_0^1 h^{c^\beta} dy\right) - \tau_s \varphi_2{}^\alpha \int_0^1 h^{c^\alpha} E_1\left(\tau_s \left|\eta' - \frac{1}{\varphi_1\varphi_2{}^\beta}\int_0^\zeta H^{i^\beta} d\zeta \Big/ \int_0^1 h^{c^\beta} dy\right|\right) d\eta'\right] \tag{38}$$

Thus, if momentum and energy transfer by transport phenomena are to be comparable with that by convection

$$\varphi_1\varphi_3 = Re \tag{39}$$

Also, Eqs. (23) and (34) imply

$$\varphi_2 = 1/h_{0_{wg}}{}^o, \quad \varphi_1 = \pi h_{0_{wg}}{}^{o^{m/2}} \varphi_3 \tag{40}$$

These particular choices for φ_1 and φ_2 imply, as we shall see, the enthalpy H^i and derivative of the stream function, $F_\zeta{}^i$, both approach unity for large ζ. Although strictly one need not include $h_{0_{wg}}{}^o$ in the expressions for φ_1 and φ_2, it is quite convenient for two reasons. First, this implies the dependent variables in the inner region have numerical values near unity which is convenient in the numerical computations, especially with regard to the enthalpy which appears in the equations raised to a high power (between about four and eight). Second, as will be demonstrated shortly, one can then readily ascertain the effect of radiative transfer in the outer, inviscid layer on the inner, boundary-layer flow. Specifically, one then obtains $\bar{\Gamma}$ as the appropriate radiation-convection energy parameter in the boundary layer which, although asymptotically of the same order of magnitude as Γ for large Re, is numerically much smaller than Γ due to $h_{0_{wg}}{}^o$ being less than unity. With this in mind, we thus chose the φ^i as

$$\varphi_1 = [\pi h_{0_{wg}}{}^{o^{m/2}} Re]^{1/2}, \quad \varphi_2 = \frac{1}{h_{0_{wg}}{}^o}, \quad \varphi_3 [Re/\pi h_{0_{wg}}{}^{o^{m/2}}]^{1/2} \tag{41}$$

thus verifying that $f_0{}^o(0)$ must indeed vanish. This expected inverse-square-root dependence of the boundary-layer thickness on Reynolds number is to be contrasted with the result obtained on the basis of an optically thin radiative transfer model, with $T_w = 0$ (see Ref. 1)

$$\varphi_{1_{\text{thin}}}[(\Gamma/\beta\mu) \log \varphi_{1_{\text{thin}}}]^\alpha = \pi Re \tag{42}$$

The source of this striking difference is the zero value of the inviscid gas enthalpy at the wall predicted by the optically thin model. Jischke[10] has shown, however, that the optically thin model is singular and that elimination of this singular behavior using the methods of singular perturbation theory implies a finite nonzero value of the inviscid gas enthalpy at the wall. Thus, as suspected elsewhere,[11] the optically thin results are not characteristic of the more general case. Letting

$$\bar{\Gamma} = (\Gamma/\pi) h_{0_{wg}}{}^{o^{\alpha+\beta-1-m/2}}, \quad \bar{\tau}_s = \tau_s h_{0_{wg}}{}^{o^{\beta-m/4}}/(\pi Re)^{1/2}$$
$$\bar{\eta} = \int_0^\zeta H^{i^\beta} d\zeta' \Big/ \int_0^1 h^{c^\beta} dy \tag{43}$$

we can write the inner, boundary-layer equations as

$$(NF^i{}_{\zeta\zeta})_\zeta + F^i F^i{}_{\zeta\zeta} = (F_\zeta{}^{i^2} - H^{i^m})/(1+j) \tag{44}$$

$$\left(\frac{N}{Pr} H_\zeta{}^i\right) + F^i H_\zeta{}^i = \bar{\Gamma} H^{i^\beta}\Big[2H^{i^\alpha} - T_w{}^4/h_{0_{wg}}{}^{o^\alpha} \times E_2(\bar{\tau}_s \bar{\eta}) - \tau_s/h_{0_{wg}}{}^{o^\alpha} \int_0^1 h^{c^\alpha} E_1(|\tau_s \eta' - \bar{\tau}_s \bar{\eta}|) d\eta'\Big] \tag{45}$$

The reduced shock layer optical depth, $\bar{\tau}_s$, can be looked upon as a Bouguer number based upon the boundary-layer thickness and indicates the relative importance of emission and self-absorption within the boundary layer. For τ_s of order unity and Re large, $\bar{\tau}_s$ will be small. Similarly, $\bar{\Gamma}$ is the reduced radiation-convection energy parameter appropriate to the boundary layer. As $h_{0_{wg}}{}^o$ is less than unity for most cases of interest (e.g. see Fig. 3) and typical values of ($\alpha + \beta - 1 - m/2$) range from five to ten, the effect of radiative transfer within the boundary layer is smaller than in the shock layer and, in many cases, is negligible.

Expanding these inner variables for large Re

$$F^i = F_0{}^i + \delta_1(Re) F_1{}^i + \delta_2(Re) F_2{}^i + \ldots$$
$$H^i = H_0{}^i + \delta_1(Re) H_1{}^i + \delta_2(Re) H_2{}^i + \ldots \tag{46}$$

Eqs. (44) and (45) can be written, to lowest order

$$(NF_{0\zeta\zeta}{}^i)_\zeta + F_{0\zeta\zeta}{}^i = (F_{0\zeta}{}^{i^2} - H_0{}^{i^m})/(1+j) \tag{47}$$

$$\left(\frac{N}{Pr} H_{0\zeta}{}^i\right)_\zeta + F_0{}^i H_{0\zeta}{}^i = \bar{\Gamma} H_0{}^{i^\beta}\Big[2H_0{}^{i^\alpha} - T_w{}^4 h_{0_{wg}}{}^{o^{-\alpha}} - \tau_s h_{0_{wg}}{}^{o^{-\alpha}} \int_0^1 h_0{}^{o^\alpha} E_1(\tau_s \eta') d\eta'\Big] \tag{48}$$

where we have made use of the fact that, to lowest order, the absorption integral is given by the outer solution. However, the condition of radiative equilibrium at the wall for the lowest-order, outer solution implies

$$T_w{}^4 + \tau_s \int_0^1 h_0{}^{o^\alpha} E_1(\tau_s \eta')\, d\eta' = 2h_{0_{wg}}{}^{o^\alpha} \tag{49}$$

Thus Eq. (48) can be rewritten

$$\left(\frac{N}{Pr} H_{0_\zeta}{}^i\right)_\zeta + F_0{}^i H_{0_\zeta}{}^i = 2\,\bar{\Gamma} H_0{}^{i^\beta}(H_0{}^{i^\alpha} - 1) \tag{50}$$

This implies that the viscous boundary layer is, to lowest or-

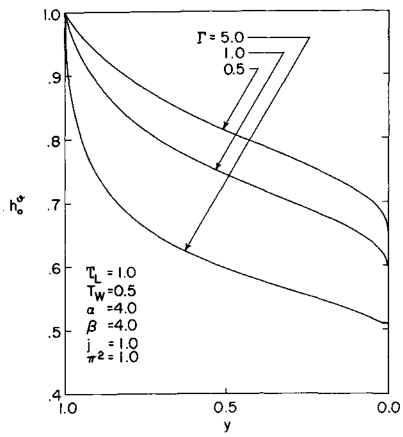

Fig. 1 Shock layer enthalpy profiles, τ_L = 1.0.

der, a region of variable emission and constant absorption with the absorption level being determined by the outer inviscid flow. Thus, as opposed to being optically thin as assumed elsewhere, absorption within the boundary layer is not negligible in that the inviscid portion of the shock layer acts as a constant external source of radiant energy providing an additional mechanism for energy transfer to the boundary layer. Also, the above is in contrast to other results[1] for the (singular) optically thin limit where the lowest-order boundary layer is a nonradiating one. The present result appears to be of general importance. Specifically, regions of small spatial extent (small such that the Bouguer number based on the thickness of the region is small) are characterized by variable emission and a constant absorption level—the constant level following from the "outer" region. Chien's[12] treatment of the radiation-structured blast wave and Cohen and Clarke's[13] analysis of the influence of viscosity on shock waves structured by radiation are other examples of this result.

In addition to satisfying boundary conditions at the wall, the lowest-order boundary-layer solution must satisfy conditions at the outer edge of the boundary layer. These conditions follow from the matching procedure. As asymptotic solutions of Eqs. (47) and (50) are

$$F_0{}^i \sim \zeta - \text{const} + o[\zeta^{-1-2\bar{\Gamma}\alpha}\exp(-\zeta^2/2)] \quad (51)$$

$$H_0{}^i \sim 1 + o[\zeta^{-2\bar{\Gamma}\alpha}\exp(-\zeta^2/2)] \quad (52)$$

the limiting matching principle gives

$$\zeta \to \infty: \quad H_0{}^i, F_{0\zeta}{}^i \to 1 \quad (53)$$

The boundary conditions at the wall for this lowest-order inner solution are

$$\zeta = 0: \quad F_0{}^i = F_{0\zeta}{}^i = 0, H_0{}^i = h_w/h_{0_{wg}}{}^o \quad (54)$$

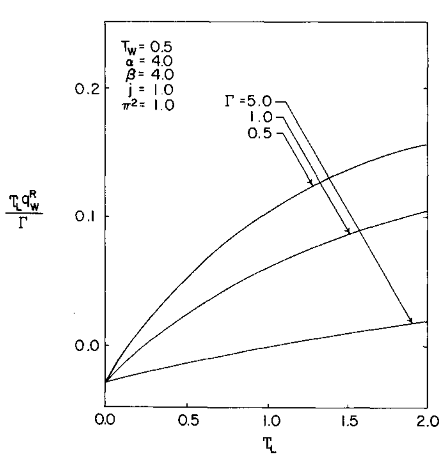

Fig. 2 Radiative transfer at the wall vs τ_L.

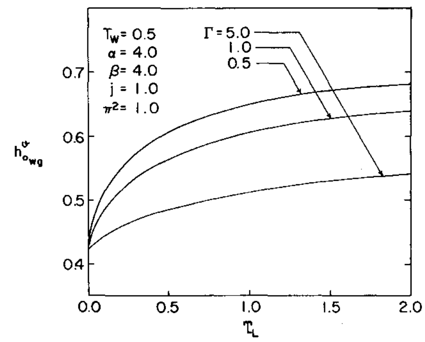

Fig. 3 Inviscid gas enthalpy at the wall vs τ_L.

The determination of higher-order contributions requires knowledge of $\delta_1(Re)$ and $\delta_2(Re)$. Comparison of the inner limit of the outer solution with the outer limit of the inner solution reveals that the two results will match if

$$\delta_1(Re) = \frac{(\log\varphi_1)^{1+j}}{\varphi_1}, \quad \delta_2(Re) = \frac{\log\varphi_1}{\varphi_1}, \quad \epsilon_1 = \frac{1}{\varphi_1} \quad (55)$$

For $j = 0$, we can eliminate either δ_1 or δ_2. However, as the axisymmetric case is of greater practical interest, we shall restrict attention to $j = 1$ in the ensuing discussion. Note that our earlier comment about the limit of $Re\epsilon_1(Re)$ for large Re is now verified. Also, the above result for $\delta_1(Re)$ is to be contrasted with that obtained for the (singular) optically thin limit

$$\delta_1(Re)_{\text{thin}} = 1/\varphi_{1\text{thin}} \quad (56)$$

The first- and second-order boundary-layer equations for the axisymmetric case can then be shown to be

$$(NF_{1\zeta\zeta}{}^i)_\zeta + F_1{}^iF_{0\zeta\zeta}{}^i + F_0{}^iF_{1\zeta\zeta}{}^i = \tfrac{1}{2}(2F_{0\zeta}{}^iF_{1\zeta}{}^i - mH_0{}^{i^{m-1}}H_1{}^i) \quad (57)$$

$$\left(\frac{N}{Pr}H_{1\zeta}{}^i\right) + F_1{}^iH_{0\zeta}{}^i + F_0{}^iH_{1\zeta}{}^i = 2\bar{\Gamma}H_0{}^{i^{\beta-1}}[(\beta+\alpha)H_0{}^{i^\alpha} - \beta)]H_1{}^i \quad (58)$$

$$(NF_{2\zeta\zeta}{}^i)_\zeta + F_2{}^iF_{0\zeta\zeta}{}^i + F_{0\zeta\zeta}{}^iF_2{}^i = \tfrac{1}{2}(2F_{0\zeta}{}^iF_{2\zeta}{}^i - mH_0{}^{i^{m-1}}H_2{}^i) \quad (59)$$

$$\left(\frac{N}{Pr}H_{2\zeta}{}^i\right)_\zeta + F_2{}^iH_{0\zeta}{}^i + F_0{}^iH_{2\zeta}{}^i = \bar{\Gamma}H_0{}^{i^{\beta-1}}\Big[2H_2{}^i\{(\beta+\alpha)H_0{}^{i^\alpha} - \beta\} + \tau_s h_{0_{wg}}{}^{o^\beta} \times \left(\int_0^1 h_0{}^\beta dy\right)^{-1}\Big\{(T_w{}^4/h_{0_{wg}}{}^{o^\alpha} - 1)\int_0^\zeta H_0{}^{i^\beta}d\zeta - \int_0^\infty(1 - H_0{}^{i^{\alpha+\beta}})d\zeta\Big\}\Big] \quad (60)$$

In deriving the second-order energy equation, the following less obvious result was obtained

$$\int_0^1 h^{c^\alpha}E_1(|\tau_s\eta' - \bar{\tau}_s\bar{\eta}|)d\eta' \sim \int_0^1 h_0{}^{o^\alpha}E_1(\tau_s\eta')d\eta' - \frac{\bar{\eta}_0}{\tau_s}h_{0_{wg}}{}^{o^\alpha}\bar{\tau}_s\ln\bar{\tau}_s + \left(\epsilon_1\log\epsilon_1 h_{0_{wg}}{}^{o^{\alpha+\beta}}\Big/\int_0^1 h_0{}^{o^\beta}dy\right)\int_0^\infty(1 - H_0{}^{i^{\alpha+\beta}})d\zeta \quad (61)$$

with errors of order ϵ_1. The lowest-order optical variable in the boundary layer, $\bar{\eta}_0$, is given by

$$\bar{\eta}_0 = \int_0^\zeta H_0{}^{i^\beta}d\zeta\Big/\int_0^1 h_0{}^{o^\beta}dy \quad (62)$$

The asymptotic matching principle gives the following boundary conditions for the first- and second-order boundary-layer

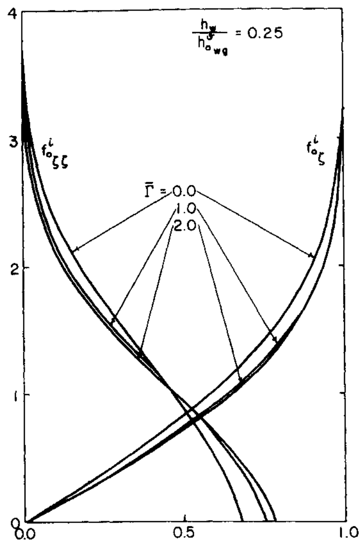

Fig. 4 Boundary - layer stream function profiles.

equations

$$\zeta \to \infty: \quad F_{1\zeta}{}^i \sim 2a_1\zeta + a_3/\pi h_{0_{wg}}{}^{o^{m/2}}, \quad H_{1\zeta}{}^i \sim 0$$

$$F_{2\zeta}{}^i \sim \frac{-2}{\pi h_{0_{wg}}{}^{o^{m/2}}} [2a_1\zeta \log\zeta + (a_1 + a_2)\zeta + a_3 \log\zeta + a_3] \quad (63)$$

$$H_{2\zeta}{}^i \sim -a_4/h_{0_{wg}}{}^o$$

At the wall, we have

$$\zeta = 0: \qquad F_1{}^i = F_{1\zeta}{}^i = H_1{}^i = 0$$
$$F_2{}^i = F_{2\zeta}{}^i = H_2{}^i = 0 \quad (64)$$

The (dimensional) convective heat-transfer and skin-friction coefficient are given by

$$q_w{}^c \frac{Pr_w}{N_w} \frac{Re^{1/2}}{\rho_s v_s h_s} = h_{0_{wg}}{}^{o^{1+m/4}} \pi^{1/2} \{H_{0\zeta}{}^i + \delta_1 H_{1\zeta}{}^i + \delta_2 H_2{}^i\}_{\zeta=0} \quad (65)$$

$$C_{f_w} \frac{A}{xN_w} Re^{1/2} = (\pi h_{0_{wg}}{}^{o^{m/2}})^{3/2} \{F_{0\zeta\zeta}{}^i + \delta_1 F_{1\zeta\zeta}{}^i + \delta_2 F_{2\zeta\zeta}{}^i\}_{\zeta=0} \quad (66)$$

with errors of order $\varphi_1{}^{-1}$. The radiative heat transfer to the body is, to order $\varphi_1{}^{-1}$, given by the lowest-order outer solution alone. Thus, the total (dimensional) heat transfer to the body is given by

$$\frac{q_w{}^R + q_w{}^c}{\rho_s v_s h_s} = \frac{\bar{\Gamma}}{\tau_L} \left(\tau_s \int_0^1 h_0{}^{o^\alpha} E_2(\tau_s \eta') d\eta' - \frac{T_w{}^4}{2} \right) + \frac{N_w}{Pr_w} \left(\frac{\pi}{Re} \right)^{1/2} h_{0_{wg}}{}^{o^{1+m/4}} (H_{0\zeta}{}^i + \delta_1 H_{1\zeta}{}^i + \delta_2 H_{2\zeta}{}^i)_{\zeta=0} \quad (67)$$

up to (but not including) order $\varphi_1{}^{-1}$.

Results and Discussion

Typical lowest-order, inviscid shock layer results are shown in Figs. 1–3. These results were obtained using the method of parametric differentiation as described elsewhere.[2] In the actual computations, the parametrically differentiated equations were solved in a manner similar to that employed in analyzing the one-dimensional, hypersonic, radiating shock layer.

The effect of the radiation-convection energy parameter, Γ, on the shock layer enthalpy profile is shown in Fig. 1 for τ_L = 1.0 and T_w = 0.5. Increasing Γ with τ_L fixed, implying a

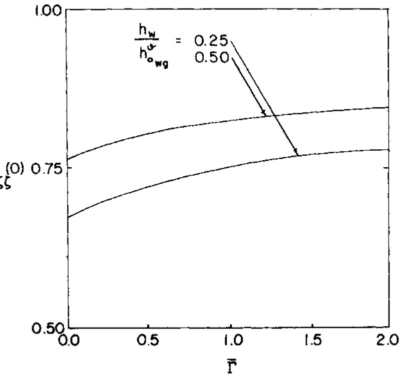

Fig. 5 Lowest-order skin-friction parameter vs $\bar{\Gamma}$.

more strongly radiating layer, leads to decreased enthalpy levels due to the associated larger energy losses. The (nondimensional) radiative heat transfer to the body

$$\frac{\tau_L}{\Gamma} q_w{}^R = \tau_s \int_0^1 h_0{}^\alpha E_2(\tau_s \eta') d\eta' - \tfrac{1}{2} T_w{}^4 \quad (68)$$

is given in Fig. 2 for various values of the Bouguer number, τ_L, and radiation-convection energy parameter, Γ. As shown, increasing τ_L at a fixed Γ, corresponding to a more strongly absorbing shock layer with higher associated enthalpy levels, leads to increased radiative heat transfer. Also, increasing Γ at a fixed τ_L yields lower radiative heat transfer due to the lower associated enthalpy levels in the shock layer. The variation of the lowest-order, inviscid gas enthalpy at the wall, $h_{0_{wg}}{}^o$, with Bouguer number τ_L for various Γ is shown in Fig. 3. Being an energy loss mechanism, radiative transfer leads to reduced values of $h_{0_{wg}}{}^o$ (the nonradiating value of $h_{0_{wg}}{}^o$ is unity). As $h_{0_{wg}}{}^o$ provides the scaling for the enthalpy level in the boundary layer, the gross effects of radiative transfer in the inviscid shock layer on the boundary-layer flow can be inferred. That is $\bar{\Gamma}$, a measure of the relative importance of energy transfer in the boundary layer by radiation as compared with convection, is given by Eq. (43), and typical values of $(\alpha + \beta - 1 - m/2)$ (e.g. from five to ten) imply, with these calculated values of $h_{0_{wg}}{}^o$, the weakly radiating nature of the boundary layer for most cases of interest.

Typical results of calculation of the boundary-layer flow are shown in Figs. 4–8. The effect of radiative transfer on the lowest-order, boundary-layer velocity profile is given in Fig. 4. As found in many other radiative gasdynamic calculations, the effect of radiative transfer here is primarily thermodynamic (as opposed to dynamic) with a relatively small in-

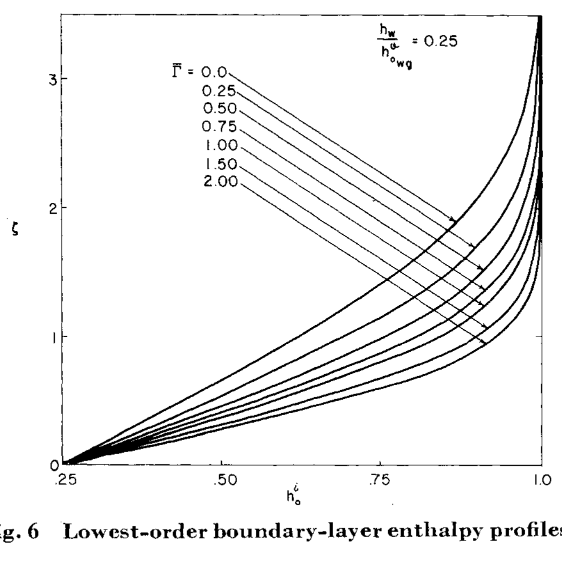

Fig. 6 Lowest-order boundary-layer enthalpy profiles.

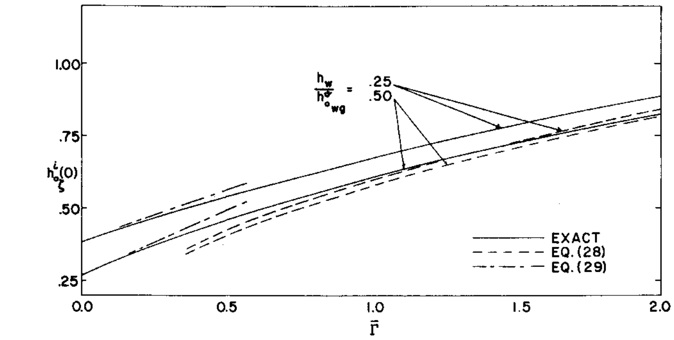

Fig. 7 Lowest-order convective heat-transfer parameter vs $\bar{\Gamma}$.

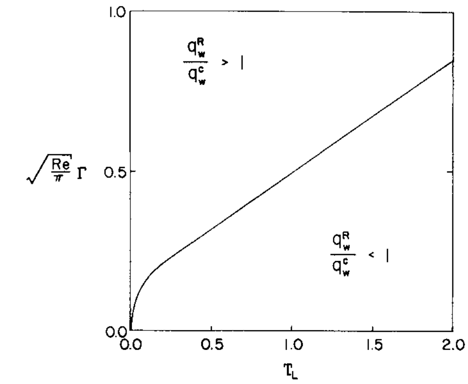

Fig. 9 Regimes of dominant radiative and convective heat transfer.

fluence on the stream function. Thus, as shown in Fig. 5 radiative transfer has a relatively small effect on the second derivative of the lowest-order stream function at the wall $F_{0\zeta\zeta}{}^i$ $(\zeta = 0)$ which is directly proportional to the skin friction coefficient, C_f. From Eq. (66), one can then infer the effect of radiative transfer on the skin-friction coefficient to be given primarily by the change in the inviscid gas enthalpy at the wall and shock detachment distance

$$C_f/C_{f\,\Gamma=0} \approx h_{0_{wg}}{}^{o^{3m}\,4}(L_{\Gamma=0}/L)^{1/2} \tag{69}$$

assuming N to be constant. Noting that the change in the shock detachment distance with the inclusion of radiative transfer is generally small, the net effect of radiative transfer is seen to be to decrease the skin friction. As m is approximately unity, Fig. 3 implies possible reductions in the skin friction by a factor of two.

A typical set of lowest-order enthalpy profiles in the boundary layer is shown in Fig. 6. As indicated, increasing Γ, corresponding to a more strongly radiating inviscid outer layer and, hence, a more strongly absorbing boundary layer, implies larger enthalpy gradients at the wall. Indeed for large $\bar{\Gamma}$, the effect of the convection of energy in the boundary layer is negligible to the lowest order in $\bar{\Gamma}$ and one can then integrate the lowest-order energy equation to obtain

$$\frac{dH_0{}^i}{d\zeta} = 2\bar{\Gamma}^{1/2}\left\{\frac{1 - H_0{}^{i\beta+1}}{\beta+1} - \frac{1 - H_0{}^{i\alpha+\beta+1}}{\alpha+\beta+1}\right\}^{1/2}, \quad \bar{\Gamma} \gg 1 \tag{70}$$

for $N = Pr = 1$. As most practical cases of interest correspond to small $\bar{\Gamma}$, it is of interest to obtain a expression for the heat transfer in this limit. To this end, we employ the formalism of the method of parametric differentiation and differentiate the lowest-order energy equation with respect to $\bar{\Gamma}$ to obtain, for $N = Pr = 1$

$$\bar{H}_{\zeta\zeta} - \bar{F}H_{0\zeta}{}^i + F_0{}^i\bar{H}_\zeta = 2H_0{}^{i\beta}(H_0{}^{i\alpha} - 1) + 2\bar{\Gamma}H_0{}^{i\beta-1}(\beta H_0{}^{i\alpha} + \alpha)\bar{H} \tag{71}$$

where

$$\bar{H} = \partial H_0{}^i/\partial\bar{\Gamma}, \quad \bar{F} = \partial F_0{}^i/\partial\bar{\Gamma} \tag{72}$$

Restricting attention to $\bar{\Gamma} = 0$ and neglecting the effect of radiative transfer on the stream function (e.g., assume $\bar{F} \approx 0$), we obtain

$$\bar{H}_\zeta(\zeta = 0) = 2\int_0^\infty d\zeta' \int_0^{\zeta'} d\zeta'' H_0{}^{i\beta}(1 - H_0{}^{i\alpha}) \exp \times \left(-\int_{\zeta'}^{\zeta''} F_0{}^i d\delta'''\right)\Big/\int_0^\infty d\zeta' \exp\left(-\int_0^{\zeta'} F_0{}^i d\zeta''\right) \tag{73}$$

which depends upon $h_w/h_{0_{wg}}{}^o$only. Evaluating the right-hand-side from exact (numerical) calculations and empirically correlating these results, we obtain

$$H_{0\zeta}{}^i(\zeta = 0) = H_{0\zeta}{}^i(\zeta = 0)_{\bar{\Gamma}=0} + \bar{\Gamma}\left[0.26 + 0.15\frac{h_w}{h_{0_{wg}}{}^o} + 0.5\left(\frac{h_w}{h_{0_{wg}}{}^o}\right)^2\right], \quad \frac{h_w}{h_{0_{wg}}{}^o} \le 0.5 \tag{74}$$

The approximate expressions for the enthalpy gradient at the wall given by Eqs. (70) and (74) along with (exact) results of numerical computation are shown in Fig. 7 for two values of $h_w/h_{0_{wg}}$.

Typical first- and second-order boundary-layer results are shown in Fig. 8 for $\bar{\Gamma} = 0.0304$, $h_w/h_{0_{wg}}{}^o = 0.796$ and $\tau_L =$

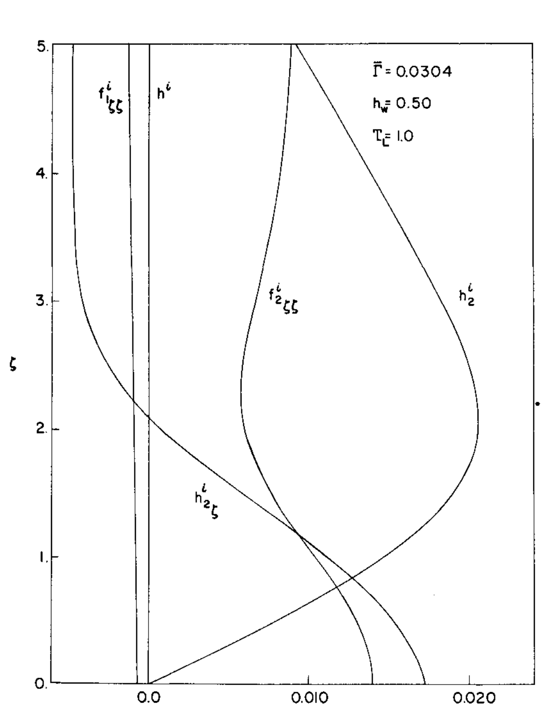

Fig. 8 First- and second-order boundary-layer profiles.

Table 1 Nondimensional velocity and enthalpy gradients in the boundary layer, $T_W = 0.5$

Γ	Γ_L	$F_{0\zeta\zeta}{}^i$ (0)	$H_{0\zeta}{}^i$ (0)	$F_{1\zeta\zeta}{}^i$ (0)	$H_{1\zeta}{}^i$ (0)	$F_{2\zeta\zeta}{}^i$ (0)	$H_{2\zeta}{}^i$ (0)
0.5	0.5	0.874	0.100	0	0	0.007	0.006
	1	0.856	0.138	−0.001	0	0.004	0.017
1.0	0.5	0.893	0.070	0	0	0.009	0.004
	1.0	0.874	0.112	−0.001	0	0.019	0.012
5.0	0.5	0.936	−0.024	0	0	0.028	−0.001
	1	0.920	0.024	0	0	0.071	0.002

1.0 corresponding to an outer inviscid flow with $\Gamma = 0.5$. In Table 1 we have tabulated first- and second-order heat-transfer and skin-friction parameters for calculations corresponding to $\Gamma = 0.5, 1.0, 5.0$ and $\tau_L = 0.5, 1.0$. In all cases, $T_w = 0.5$. As indicated, the first- and second-order contributions to the heat transfer and skin friction are quite small and thus, while of theoretical interest, are practically negligible. This is due to the smallness of $\bar{\Gamma}$ which, in turn, results from the reduction of $h_{0_{wg}}{}^o$ below its nonradiating value of unity.

As one expects $\bar{\Gamma}$ to be small for most cases of interest, one could sensibly assume $\bar{\Gamma}$ to vanish within the boundary layer. In this approximation, radiative transfer affects the skin friction and convective heat transfer only through the scaling of the enthalpy and stream function in the boundary layer (neglecting the change in the shock detachment distance and hence the Reynolds number). Eq. (69) gives the effect of radiative transfer on the skin friction. The convective heat transfer is, in this approximation, given by

$$\frac{q_w{}^c}{\rho_s v_s h_s}\left(\frac{Re}{\pi}\right)^{1/2} \approx h_{0_{wg}}{}^{o^{1+m/4}} G\left(\frac{h_w}{h_{0_{wg}}{}^o}\right) \tag{75}$$

assuming $N = Pr = 1$. The function G follows from the nonradiating boundary-layer solution[14] and, for $N = Pr = 1$, one can empirically correlate the numerical results to obtain

$$G(h_w/h_{0_{wg}}{}^o) \approx 0.50(1 - h_w/h_{0_{wg}}{}^o) \tag{76}$$

Using these approximations, the convective heat transfer (times $Re^{1/2}$) has been calculated for $N = Pr = \pi^2 = 1.0$ for various values of the radiative parameters $\bar{\Gamma}$ and τ_L. Comparing these results with those of Fig. 2, we see that, unless Γ is small, the convective heat transfer is generally less than the radiative heat transfer in the high Reynolds number limit for which the present calculations are applicable.

Assuming Γ small, the convective and radiative components of the heat transfer can be determined analytically and compared to determine the locus of conditions which delinate the regime where radiative heat transfer exceeds convective heat transfer in the high Reynolds number limit. For Γ small, the radiative heat transfer to the body is given, to lowest order by the non-radiating enthalpy profile. Thus

$$\frac{q_w{}^R}{\rho_s v_s h_s} = \frac{\Gamma}{2\tau_L}\left[1 - 2E_3(\tau_L) - T_w{}^4\right], \quad \Gamma \ll 1 \tag{77}$$

with the gas enthalpy at the wall being given by

$$h_{0_{wg}}{}^o = [\tfrac{1}{2}(1 - E_2(\tau_L) + T_w{}^4)]^{1/\alpha}, \quad \Gamma \ll 1 \tag{78}$$

Thus, for $N = Pr = 1$, the convective heat transfer follows from Eqs. (75) and (76) is given by

$$\frac{q_w{}^c}{\rho_s v_s h_s} \approx \frac{1}{4}\left(\frac{\pi}{Re}\right)^{1/2}\left[\frac{1}{2}(1 - E_2(\tau_L) + T_w{}^4)\right]^{(4+m/4\alpha)} \times \left[1 - T_w\left(\frac{2}{1 - E_2(\tau_L) + T_w{}^4}\right)^{1/\alpha}\right] \tag{79}$$

using these small Γ results, the locus of points in a $[\tau_L, \Gamma(Re)/\pi)^{1/2}]$ plane for which the radiative and convective components of the heat transfer are equal is shown in Fig. 9 for $T_w = 0.0$. These results indicate that in the high Reynolds number limit for which the present analysis is valid, the radiative component of the heat transfer dominates the convective heating at rather small values of Γ (Γ of order $Re^{-1/2}$).

The extension of the present results to include the effect of nongray radiative transfer is, to lowest order, straight forward. The outer (inviscid) energy equation becomes

$$f_0{}^o h_{0_g}{}^o = \Gamma\left[2\varkappa_p h_0{}^{o\alpha} - \int_0^\infty \varkappa_\nu E_2(\tau_\nu)\bar{B}_\nu(T_w)d\nu - \int_0^\infty \varkappa_\nu \int_0^{\tau_{\nu_s}} \bar{B}_\nu(T_0{}^o)E_1(|\tau_\nu - t|)dt\cdot d\nu\right] \tag{80}$$

where the nondimensional Planck function, $\bar{B}_\nu$, is

$$\bar{B}_\nu = \frac{\pi h\nu^3}{\sigma T_s{}^4 c^2}\frac{1}{e^{h\nu/kT} - 1} \tag{81}$$

The Planck mean absorption coefficient, $\varkappa_p$, is

$$\varkappa_p = \frac{\pi}{\sigma T^4}\int_0^\infty \varkappa_\nu \bar{B}_\nu(T)d\nu \tag{82}$$

and can be reasonably assumed to be proportional to some power (β) of the enthalpy. The spectral absorption coefficient $\varkappa_\nu$ is here normalized by the value of the Planck mean absorption coefficient at the shock. The boundary-layer scaling with Reynolds number is the same as earlier with the lowest-order boundary-layer energy equation being

$$\left(\frac{N}{Pr}H_{0\zeta}{}^i\right)_\zeta + F_0{}^i H_{0\zeta}{}^i = \left(\frac{\Gamma}{\pi}h_{0_{wg}}{}^{o^{\alpha-1-m/2}}\varkappa_{p_{wg}}\right) \times \left[2\varkappa_p{}^i H_0{}^{i\alpha} - \varkappa_p{}^i\frac{T_w{}^4}{h_{0_{wg}}{}^{o\alpha}} - \int_0^\infty \frac{\varkappa_\nu}{\varkappa_{p_{wg}}}\int_0^{\tau_{\nu_s}}\frac{\bar{B}_\nu(T_0{}^o)}{h_{0_{wg}}{}^{o\alpha}}E_1(t)dtd\nu\right] \tag{83}$$

which is similar to Eq. (50) in that the boundary layer is a region of variable emission and constant absorption with the constant absorption level being determined by the outer inviscid flow. Also, the boundary layer in the nongray case is weakly radiating as a result of the reduced inviscid gas enthalpy at the wall provided $\varkappa_\nu$, the spectral absorption coefficient, is not so large that $h_{0_{wg}}{}^{o\alpha-1-m/2}\varkappa_\nu$ is of order unity. In that case, the boundary layer would be weakly emitting and strongly absorbing corresponding, perhaps, to a situation of practical interest. The lowest-order, outer (inviscid) and inner (boundary-layer) energy equations, Eqs. (80) and (83) both assume $\rho\kappa_\nu L\, Re^{-1/2}$ to be small and have errors of the order of

$$Re^{-1/2}\log(Re)\int_0^\infty (\rho\kappa_\nu L)^2 B_\nu d_\nu$$

which may not be small if there are very strongly absorbing lines for which $\varkappa_\nu$ is quite large and/or if there is an extensive optically thick region in which significant energy transfer occurs. If these quantities are not small, then self-absorption within the boundary layer is not negligible and the inviscid outer region and viscous inner region are coupled, radiatively, to lowest order and we have a strong interaction problem. The data given by Biberman, et al.[15] for $T = 6000°K$ and $p = 1.0$ atm. suggest that at very high frequencies $\rho\kappa_\nu$ is of the order of 10. cm^{-1} implying the present analysis to be valid if the boundary-layer thickness is less than aproximately 0.01 cm corresponding, crudely, to a shock detachment distance of 30. cm at a Reynolds number of 10^7.

Appendix A

$$a_1 = -(\pi/8)h_{0_{wg}}{}^{o^{m/2-1}}ma_4, \quad j = 1$$
$$(\pi/2)h_{0_{wg}}{}^{o^{m/2-1}}ma_4, \quad j = 0 \tag{A1}$$

$$a_2 = (m\pi/8)h_{0_{wg}}{}^{o^{m/2-1}}(-2a_5 + 3a_4), \quad j = 1$$
$$(m\pi/4)h_{0_{wg}}{}^{o^{m/2-1}}(2a_5 - a_4), \quad j = 0 \tag{A2}$$

$$a_3 a^o = -ma_4/4h_{0_{wg}}{}^o f_{1_{wg}}{}^o, \quad j = 1$$
$$ma_4/2h_{0_{wg}}{}^o f_{1_{wg}}{}^o + \pi h_{0_{wg}}{}^{o^{m/2-1}}(ma_6/2), \quad j = 0 \tag{A3}$$

$$a_4 = -\tau_s\Gamma h_{0_{wg}}{}^{o^{2\beta}}(T_w{}^4 - h_{0_{wg}}{}^{o\alpha})/(\pi h_{0_{wg}}{}^{o^{m/2}} - 2\Gamma\alpha h_{0_{wg}}{}^{o^{\alpha+\beta-1}}) \times \int_0^1 h_0{}^{o\beta}dy \tag{A4}$$

$$a_5 = a_4\left[\gamma + \log\left(\tau_s h_{0_{wg}}{}^{o\beta}\Big/\int_0^1 h_0{}^{o\beta}dy\right) - E_1(\tau_s) - \tau_s\int_0^1\frac{dh_0{}^{o\alpha}}{d\eta'}E_1(\tau_s\eta')d\eta'\Big/T_w{}^4 - h_{0_{wg}}{}^{o\alpha}\right] \tag{A5}$$

$$a_6 = \frac{1}{2\alpha h_{0_{wg}}{}^{o\,\alpha-1}}\left[\frac{a_4 f_{1_{wg}}{}^{o}}{\Gamma h_{0_{wg}}{}^{o\beta}} - \tau_L h_{0_{wg}}{}^{o\,\alpha+\beta}\int_0^\infty (1 - H_0{}^{i\,\alpha+\beta})d\zeta'\right] \tag{A6}$$

$$a^7 = 1/2\alpha h_{0_{wg}}{}^{o\,\alpha-1}\Big[\pi h_{0_{wg}}{}^{o\,m/2} a_6 + f_{1_{iwg}}{}^{o} a_5/\Gamma h_{0_{wg}}{}^{o\beta} + \tau_s\beta\int_0^1 h_0{}^{o\alpha} e^{-\tau_s\eta_0'}\frac{\eta_1'}{\eta_0'}d\eta_0' + \tau_s(\alpha+\beta)\int_0^1 h_0{}^{o\,\alpha-1} h_1{}^{o} E_1 \times (\tau_s\eta_0')d\eta_0' + \tau_L h_{0_{wg}}{}^{o\,\alpha+\beta}(\gamma + \log(\tau_L h_{0_{wg}}{}^{o\beta})) \times \int_0^\infty (1 - H_0{}^{i\,\alpha+\beta})d\zeta'\Big] \tag{A7}$$

where γ is the Euler constant, 0.577 ....

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